Equivalence of Two-Loop Superstring Amplitudes in the Pure Spinor and RNS Formalisms

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The pure spinor formalism for the superstring has recently been used to compute massless four-point two-loop amplitudes in a manifestly super-Poincaré covariant manner. In this paper, we show that when all four external states are Neveu-Schwarz, the two-loop amplitude coincides with the RNS result.

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1. Introduction

String theory is currently the most promising model for unification of the forces. In bosonic string theory, the prescription for computing perturbative scattering amplitudes is well-developed and has been used to compute amplitudes with arbitrary numbers of loops. Unfortunately, these multiloop amplitudes suffer from unphysical divergences which make bosonic string theory inconsistent. In superstring theory, spacetime supersymmetry helps in cancelling these divergences. However, because spacetime supersymmetry is not manifest in the Ramond-Neveu-Schwarz (RNS) formalism [1] for the superstring, it is difficult to explicitly prove the cancellation of divergences using this formalism. Although the Green-Schwarz (GS) formalism [2] for the superstring is manifestly spacetime supersymmetric, its non-quadratic action makes it difficult to quantize except in light-cone gauge.

Five years ago, a new formalism for the superstring with manifest spacetime supersymmetry was introduced which uses pure spinors as worldsheet ghosts [3]. Since the worldsheet action is quadratic, it is straightforward to compute manifestly super-Poincaré covariant N-point tree amplitudes using this formalism and, last year, it was shown how to compute multiloop amplitudes [4]. In addition to proving various vanishing theorems related to perturbative finiteness and S-duality [4], super-Poincaré covariant massless four-point one-loop [4] and two-loop [5] amplitudes were explicitly computed.

To check consistency of the new formalism, it is useful to compare these amplitudes with those amplitudes that have also been computed using the RNS and GS formalisms. For massless N-point tree amplitudes involving four or fewer Ramond states and an arbitrary number of Neveu-Schwarz states, the equivalence with the RNS computation was proven in [6]. And for massless four-point one-loop amplitudes, the equivalence with the RNS and GS computations was proven in [7].

For massless four-point two-loop amplitudes, computations have only been performed using the RNS formalism for the case when all four external states are Neveu-Schwarz [8] [9]. Because of the need to sum over spin structures and include surface term contributions, these RNS computations are extremely complicated. On the other hand, computation of massless four-point two-loop amplitudes using the super-Poincaré covariant formalism is easy since the fermionic worldsheet variables only contribute through their zero modes[5]. The final result is quite simple and is expressed as a superspace integral in terms of the ten-dimensional super-Yang-Mills and supergravity superfields.

In this paper, the integral over superspace will be explicitly performed for the case when all external states are in the Neveu-Schwarz sector. The amplitude will then be shown to coincide with the RNS result of [8][9].

2. Comparison of Two-Loop Amplitudes

As derived in [5] using the methods of [4], the four-point two-loop Type IIB amplitude computed using the pure spinor formalism is

$$\mathcal{A} = \int d^2\Omega_{11}d^2\Omega_{12}d^2\Omega_{22}\prod_{R=1}^{4}\int d^2 z_R \frac{\exp\left(-\Sigma_{R,S=1}^4 k_R \cdot k_S G(z_R, z_S)\right)}{(\det Im\Omega)^5} \tag{2.1}$$

$$\left| \left(\int d^5\theta\right)^{\alpha\beta\gamma}(\gamma^{mnpqr})_{\alpha\beta}\gamma_{\gamma\delta}^s \right.$$

$$\left. \left(\mathcal{F}_{mn}^1(\theta)\mathcal{F}_{pq}^2(\theta)\mathcal{F}_{rs}^3(\theta)W^{4\delta}(\theta)\Delta(z_1, z_3)\Delta(z_2, z_4) + \text{perm}(1234)\right) \right|^2$$

where Ω_{CD} is the genus-two period matrix for $C, D = 1$ to 2, $\Delta(y, z) = \epsilon^{CD}\omega_C(y)\omega_D(z)$, ω_C are the two holomorphic one-forms, $G(y, z)$ is the scalar Green's function, $|\ |^2$ denotes the product of left and right-moving open superstring expressions, $W^{R\alpha}(\theta)$ and $\mathcal{F}_{mn}^R(\theta)$ are the linearized spinor and vector super-Yang-Mills superfield-strengths for the Rth external state with momentum k_R^m satisfying $k_R \cdot k_R = 0$,

$$\left(\int d^5\theta\right)^{\alpha\beta\gamma} = (T^{-1})_{\rho_1...\rho_{11}}^{\alpha\beta\gamma}\epsilon^{\rho_1...\rho_{16}}\frac{\partial}{\partial\theta^{\rho_{12}}}\cdots\frac{\partial}{\partial\theta^{\rho_{16}}}, \tag{2.2}$$

and $(T^{-1})_{\rho_1...\rho_{11}}^{\alpha\beta\gamma}$ is a Lorentz-invariant tensor which is antisymmetric in $[\rho_1...\rho_{11}]$ and symmetric and γ-matrix traceless in $(\alpha\beta\gamma)$. Up to an overall normalization constant,

$$(T^{-1})_{\rho_1...\rho_{11}}^{\alpha\beta\gamma} = \epsilon_{\rho_1...\rho_{16}}(\gamma^m)^{\kappa\rho_{12}}(\gamma^n)^{\sigma\rho_{13}}(\gamma^p)^{\tau\rho_{14}}(\gamma_{mnp})^{\rho_{15}\rho_{16}}(\delta_\kappa^{(\alpha}\delta_\sigma^\beta\delta_\tau^{\gamma)} - \frac{1}{40}\gamma_q^{(\alpha\beta}\delta_\kappa^{\gamma)}\gamma_{\sigma\tau}^q). \tag{2.3}$$

Comparing (2.1) with the RNS result of [8][9] and ignoring the Ramond component fields in the superfields $W^{R\alpha}$ and $\mathcal{F}_{mn}^R$, one finds that the results coincide if

$$t_8^{m_1 n_1...m_4 n_4}F_{m_1 n_1}^1 F_{m_2 n_2}^2 F_{m_3 n_3}^3 F_{m_4 n_4}^4 \mathcal{Y} = \left(\int d^5\theta\right)^{\alpha\beta\gamma}(\gamma^{mnpqr})_{\alpha\beta}\gamma_{\gamma\delta}^s \tag{2.4}$$

$$\left(\mathcal{F}_{mn}^1(\theta)\mathcal{F}_{pq}^2(\theta)\mathcal{F}_{rs}^3(\theta)W^{4\delta}(\theta)\Delta(z_1, z_3)\Delta(z_2, z_4) + \text{perm}(1234)\right),$$

where $t_8 F^1 F^2 F^3 F^4$ is the well-known kinematic factor appearing also in four-point tree-level and one-loop computations, F_{mn}^R is the ordinary linearized Yang-Mills field-strength of the Rth external state, and

$$\mathcal{Y} = (k_1 - k_2) \cdot (k_3 - k_4)\Delta(z_1, z_2)\Delta(z_3, z_4) \tag{2.5}$$

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$$+(k_1 - k_3) \cdot (k_2 - k_4)\Delta(z_1, z_3)\Delta(z_2, z_4) + (k_1 - k_4) \cdot (k_2 - k_3)\Delta(z_1, z_4)\Delta(z_2, z_3).$$

To evaluate the right-hand side of (2.4), it is convenient to use the notation

$$(T^{-1})^{\alpha\beta\gamma}{}_{\rho_1...\rho_{11}} \epsilon^{\rho_1...\rho_{16}} \longrightarrow \langle \lambda^\alpha \lambda^\beta \lambda^\gamma \theta^{\rho_{12}} \theta^{\rho_{13}} \theta^{\rho_{14}} \theta^{\rho_{15}} \theta^{\rho_{16}} \rangle \tag{2.6}$$

where λ^α is a pure spinor, which is motivated by the original definition of $(T^{-1})^{\alpha\beta\gamma}{}_{\rho_1...\rho_{11}}$ in the amplitude computations of [3]. Using that $\frac{\partial}{\partial\theta^\alpha}$ can be substituted by $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m \theta)_\alpha \partial_m$ because of conservation of momentum, the right-hand side of (2.4) can be written as

$$\Delta(z_1, z_3)\Delta(z_2, z_4)\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s)_\delta(\theta^5)^{[\rho_1...\rho_5]}\rangle \ (D^5)_{[\rho_1...\rho_5]} \left(\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}W^{4\delta}\right) \tag{2.7}$$

$$+ \quad \text{permutations} \quad \text{of} \quad (1234).$$

Note that throughout this paper, we will use the antisymmetrization convention that

$$f_{[a_1...a_N]} = \frac{\sum_{\text{perm}(1...N)}(-1)^{sign(\sigma)}f_{a_{\sigma(1)}...a_{\sigma(N)}}}{N!}. \tag{2.8}$$

Since we only want to consider the Neveu-Schwarz sector and $\mathcal{F}_{mn}$ is bosonic while W^α is fermionic, the only contribution to this computation comes from terms in which an even number of D's act upon each $\mathcal{F}$ and an odd number of D's act on W. One therefore has

$$D^5(\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}W^{4\delta}) = \mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs} \ D^5 W^{4\delta} + \tag{2.9}$$

$$5[(D^4\mathcal{F}^1_{mn})\mathcal{F}^2_{pq}\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}(D^4\mathcal{F}^2_{pq})\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}\mathcal{F}^2_{pq}(D^4\mathcal{F}^3_{rs})] \ DW^{4\delta}$$

$$+10[(D^2\mathcal{F}^1_{mn})\mathcal{F}^2_{pq}\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}(D^2\mathcal{F}^2_{pq})\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}\mathcal{F}^2_{pq}(D^2\mathcal{F}^3_{rs})] \ D^3 W^{4\delta}$$

$$+30[(D^2\mathcal{F}^1_{mn})(D^2\mathcal{F}^2_{pq})\mathcal{F}^3_{rs} + (D^2\mathcal{F}^1_{mn})\mathcal{F}^2_{pq}(D^2\mathcal{F}^3_{rs}) + \mathcal{F}^1_{mn}(D^2\mathcal{F}^2_{pq})(D^2\mathcal{F}^3_{rs})] \ DW^{4\delta},$$

where the spinor indices on the five D's are antisymmetrized and the combinatoric factors in (2.9) come from the different ways of splitting up these five indices.

After using $D_\alpha W^\delta = \frac{1}{4}(\gamma^{tu})_\alpha{}^\delta \mathcal{F}_{tu}$, (2.7) is proportional to

$$\Delta(z_1, z_3)\Delta(z_2, z_4)\langle(\lambda\gamma^{mnpq[r}\lambda)(\lambda\gamma^{s]}\gamma^{tu}\theta)(\theta)^4\rangle \left[\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs} \ D^4\mathcal{F}^4_{tu} + \right. \tag{2.10}$$

$$5[(D^4\mathcal{F}^1_{mn})\mathcal{F}^2_{pq}\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}(D^4\mathcal{F}^2_{pq})\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}\mathcal{F}^2_{pq}(D^4\mathcal{F}^3_{rs})] \ \mathcal{F}^4_{tu}$$

$$+10[(D^2\mathcal{F}^1_{mn})\mathcal{F}^2_{pq}\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}(D^2\mathcal{F}^2_{pq})\mathcal{F}^3_{rs} + \mathcal{F}^1_{mn}\mathcal{F}^2_{pq}(D^2\mathcal{F}^3_{rs})] \ D^2\mathcal{F}^4_{tu}$$

$$+30[(D^2\mathcal{F}^1_{mn})(D^2\mathcal{F}^2_{pq})\mathcal{F}^3_{rs} + (D^2\mathcal{F}^1_{mn})\mathcal{F}^2_{pq}(D^2\mathcal{F}^3_{rs}) + \mathcal{F}^1_{mn}(D^2\mathcal{F}^2_{pq})(D^2\mathcal{F}^3_{rs})]\ \mathcal{F}^4_{tu}\Big]$$

$$+\quad \text{permutations}\quad \text{of}\quad (1234),$$

where the spinor indices on the four D's are antisymmetrized and contracted with the spinor indices on $(\theta)^4$. As will be explained later, all terms in (2.10) containing factors of $D^4\mathcal{F}$ will not contribute to the amplitude.

Using the relations $D_\alpha\mathcal{F}^{mn} = 2k^{[m}\gamma^{n]}_{\alpha\beta}W^\beta$ and $D_\beta W^\gamma = \frac{1}{4}(\gamma^{mn})_\beta^{\ \gamma}\mathcal{F}_{mn}$ where k^m is the momentum, one can express $D^2\mathcal{F}_{mn}$ and $D^4\mathcal{F}_{mn}$ in terms of $\mathcal{F}_{mn}$ as

$$D_\beta D_\alpha \mathcal{F}_{mn} = -\frac{1}{2}k_{[m}\left(\gamma_{n]}\gamma^{tu}\right)_{\alpha\beta}\mathcal{F}_{tu}, \tag{2.11}$$

$$D_\delta D_\gamma D_\beta D_\alpha \mathcal{F}_{mn} = \frac{1}{4}k_{[m}\left(\gamma_{n]}\gamma^{tu}\right)_{\alpha\beta}k_t\left(\gamma_u\gamma^{vw}\right)_{\gamma\delta}\mathcal{F}_{vw}.$$

Plugging (2.11) into (2.10) and replacing $\mathcal{F}^R_{mn}$ with its $\theta = 0$ component F^R_{mn}, one obtains that the right-hand side of (2.4) is proportional to

$$\Delta(z_1, z_3)\Delta(z_2, z_4)\langle(\lambda\gamma^{mnpq[r}\lambda)(\lambda\gamma^{s]}\gamma^{tu}\theta)(\theta\gamma^{fgh}\theta)(\theta^{jkl}\theta)\rangle \tag{2.12}$$

$$\Big[k_t^4 k_g^4 \eta_{hj}\eta_{uf}F^1_{mn}F^2_{pq}F^3_{rs}F^4_{kl}+$$

$$5\eta_{hj}[k_m^1 k_g^1 \eta_{nf}F^1_{kl}F^2_{pq}F^3_{rs} + k_p^2 k_g^2 \eta_{qf}F^1_{mn}F^2_{kl}F^3_{rs} + k_r^3 k_g^3 \eta_{sf}F^1_{mn}F^2_{pq}F^3_{kl}]\ F^4_{tu}$$

$$+10k_t^4\eta_{uj}[k_m^1\eta_{nf}F^1_{gh}F^2_{pq}F^3_{rs} + k_p^2\eta_{qf}F^1_{mn}F^2_{gh}F^3_{rs} + k_r^3\eta_{sf}F^1_{mn}F^2_{pq}F^3_{gh}]\ F^4_{kl}$$

$$+30[k_m^1 k_p^2 \eta_{nf}\eta_{qj}F^1_{gh}F^2_{kl}F^3_{rs} + k_m^1 k_r^3\eta_{nf}\eta_{sj}F^1_{gh}F^2_{pq}F^3_{kl}$$

$$+k_p^2 k_r^3 \eta_{qf}\eta_{sj}F^1_{mn}F^2_{gh}F^3_{kl}]\ F^4_{tu}\Big]$$

$$+\quad \text{permutations}\quad \text{of}\quad (1234).$$

To check if (2.12) reproduces the desired $t_8 F^1 F^2 F^3 F^4$ contractions, one needs to evaluate

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s\gamma^{tu}\theta)(\theta\gamma^{fgh}\theta)(\theta^{jkl}\theta)\rangle = \tag{2.13}$$

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{stu}\theta)(\theta\gamma^{fgh}\theta)(\theta^{jkl}\theta)\rangle + 2\langle(\lambda\gamma^{mnpqr}\lambda)\eta^{s[t}(\lambda\gamma^{u]}\theta)(\theta\gamma^{fgh}\theta)(\theta^{jkl}\theta)\rangle.$$

Fortunately, the properties of pure spinors and the symmetries of (2.13) make this a straightforward task. Since (2.13) contains fourteen vector indices and is Lorentz invariant, it can be expressed in terms of linear combinations of products of seven η_{pq} tensors, or products of one ten-dimensional ϵ tensor and two η_{pq} tensors. However, since the

four-point amplitude only involves three independent momenta and four polarizations, the ten-dimensional ϵ tensor cannot contribute to the four-point amplitude. One can easily check that the only possible linear combination of η_{pq} tensors which has the appropriate symmetries is

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{stu}\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle = \tag{2.14}$$

$$X\left[\delta^{[s}_{[f}\delta^t_g\eta^{u][m}\delta^n_{h]}\delta^p_{[j}\delta^q_k\delta^{r]}_{l]} + \delta^{[s}_{[j}\delta^t_k\eta^{u][m}\delta^n_{l]}\delta^p_{[f}\delta^q_g\delta^{r]}_{h]}\right.$$

$$\left. - A\eta^{v[s}\delta^t_{[f}\eta^{u][m}\delta^n_g\eta_{h][j}\delta^p_k\delta^q_{l]}\delta^{r]}_v - A\eta^{v[s}\delta^t_{[j}\eta^{u][m}\delta^n_k\eta_{l][f}\delta^p_g\delta^q_{h]}\delta^{r]}_v\right],$$

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^u\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle = Z\left[\delta^{[m}_{[j}\delta^n_k\delta^p_{l]}\delta^q_{[f}\delta^{r]}_g\delta^u_{h]} + \delta^{[m}_{[f}\delta^n_g\delta^p_{h]}\delta^q_{[j}\delta^{r]}_k\delta^u_{l]}\right. \tag{2.15}$$

$$\left. - B\delta^{[m}_{[j}\delta^n_k\eta_{l][f}\delta^p_g\delta^q_{h]}\eta^{r]u} - B\delta^{[m}_{[f}\delta^n_g\eta_{h][j}\delta^p_k\delta^q_{l]}\eta^{r]u}\right],$$

where A, B, X and Z are constants. The coefficients A and B are determined from the pure spinor conditions

$$\eta_{ms}\eta_{nt}(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{stu}\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta) = 0, \tag{2.16}$$

$$\eta_{mu}(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^u\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta) = 0 \tag{2.17}$$

to be $A = 1$ and $B = \frac{1}{2}$. And the constants X and Z are determined to be $X = 3Z = -\frac{12}{35}$ from the relation

$$(\lambda\gamma^{mnpqr}\lambda)(\theta\gamma_{npq}\theta) = 96(\lambda\gamma^m\theta)(\lambda\gamma^r\theta)$$

and the normalization condition that

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1.$$

Note that (2.14) and (2.15) imply that

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{stu}\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle\eta^{hj} = \langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^u\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle\eta^{hj} = 0, \tag{2.18}$$

so there is no contribution from the second and third lines of (2.12) which come from terms in (2.10) with a $D^4\mathcal{F}$ factor.

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Using the above formulæ, it is straightforward to evaluate (2.12) with the help of the mathematica package GAMMA [10] for performing the tedious sum over the antisymmetrized deltas.[3] Writing $F_{mn}^R = k_m^R e_n^R - k_n^R e_m^R$ where e_m^R is the polarization tensor satisfying $\eta^{mn} k_m^R e_n^R = 0$, and summing over all permutations of the (1234) indices, one obtains an expression containing approximately 250 terms. Using momentum conservation and expressing contractions of momenta in terms of the Mandelstam variables $s = -2(k^1 \cdot k^2)$, $t = -2(k^2 \cdot k^3)$ and $u = -2(k^1 \cdot k^3)$, one obtains that the right-hand side of (2.4) is proportional to $\Delta(z_1, z_2)\Delta(z_3, z_4)$ multiplied by

$$+2(k^2 \cdot e^3)(k^2 \cdot e^4)(e^1 \cdot e^2)t^2 + 2(k^2 \cdot e^4)(k^4 \cdot e^3)(e^1 \cdot e^2)t^2 \tag{2.19}$$

$$-2(k^2 \cdot e^4)(k^3 \cdot e^2)(e^1 \cdot e^3)t^2 + 2(k^3 \cdot e^4)(k^4 \cdot e^2)(e^1 \cdot e^3)t^2$$

$$-2(k^2 \cdot e^3)(k^4 \cdot e^2)(e^1 \cdot e^4)t^2 - 2(k^4 \cdot e^2)(k^4 \cdot e^3)(e^1 \cdot e^4)t^2$$

$$+2(k^2 \cdot e^4)(k^3 \cdot e^1)(e^2 \cdot e^3)t^2 + 2(k^2 \cdot e^3)(k^4 \cdot e^1)(e^2 \cdot e^4)t^2$$

$$+2(k^3 \cdot e^1)(k^4 \cdot e^3)(e^2 \cdot e^4)t^2 + 2(k^4 \cdot e^1)(k^4 \cdot e^3)(e^2 \cdot e^4)t^2$$

$$-2(k^3 \cdot e^1)(k^4 \cdot e^2)(e^3 \cdot e^4)t^2 + 2(k^2 \cdot e^3)(k^3 \cdot e^4)(e^1 \cdot e^2)tu$$

$$-2(k^2 \cdot e^4)(k^4 \cdot e^3)(e^1 \cdot e^2)tu - 2(k^3 \cdot e^2)(k^3 \cdot e^4)(e^1 \cdot e^3)tu$$

$$-2(k^3 \cdot e^4)(k^4 \cdot e^2)(e^1 \cdot e^3)tu + 2(k^3 \cdot e^2)(k^4 \cdot e^3)(e^1 \cdot e^4)tu$$

$$+2(k^4 \cdot e^2)(k^4 \cdot e^3)(e^1 \cdot e^4)tu + 2(k^3 \cdot e^1)(k^3 \cdot e^4)(e^2 \cdot e^3)tu$$

$$+2(k^3 \cdot e^4)(k^4 \cdot e^1)(e^2 \cdot e^3)tu - 2(k^3 \cdot e^1)(k^4 \cdot e^3)(e^2 \cdot e^4)tu$$

$$-2(k^4 \cdot e^1)(k^4 \cdot e^3)(e^2 \cdot e^4)tu - 2(k^3 \cdot e^2)(k^4 \cdot e^1)(e^3 \cdot e^4)tu$$

$$+2(k^3 \cdot e^1)(k^4 \cdot e^2)(e^3 \cdot e^4)tu - 2(k^2 \cdot e^3)(k^2 \cdot e^4)(e^1 \cdot e^2)u^2$$

$$-2(k^2 \cdot e^3)(k^3 \cdot e^4)(e^1 \cdot e^2)u^2 + 2(k^2 \cdot e^4)(k^3 \cdot e^2)(e^1 \cdot e^3)u^2$$

$$+2(k^3 \cdot e^2)(k^3 \cdot e^4)(e^1 \cdot e^3)u^2 + 2(k^2 \cdot e^3)(k^4 \cdot e^2)(e^1 \cdot e^4)u^2$$

$$-2(k^3 \cdot e^2)(k^4 \cdot e^3)(e^1 \cdot e^4)u^2 - 2(k^2 \cdot e^4)(k^3 \cdot e^1)(e^2 \cdot e^3)u^2$$

[3] We are very greatful to Dr. Ulf Gran, the author of the GAMMA package, for providing by request an efficient function to expand the antisymmetrized deltas, which is not contained in the version available to download at http://www.mth.kcl.ac.uk/~ugran/.

$$-2(k^3 \cdot e^1)(k^3 \cdot e^4)(e^2 \cdot e^3)u^2 - 2(k^3 \cdot e^4)(k^4 \cdot e^1)(e^2 \cdot e^3)u^2$$

$$-2(k^2 \cdot e^3)(k^4 \cdot e^1)(e^2 \cdot e^4)u^2 + 2(k^3 \cdot e^2)(k^4 \cdot e^1)(e^3 \cdot e^4)u^2$$

$$+(e^1 \cdot e^2)(e^3 \cdot e^4)t^2u - (e^1 \cdot e^4)(e^2 \cdot e^3)t^2u$$

$$-(e^1 \cdot e^3)(e^2 \cdot e^4)t^3 + (e^1 \cdot e^3)(e^2 \cdot e^4)tu^2$$

$$-(e^1 \cdot e^2)(e^3 \cdot e^4)tu^2 + (e^1 \cdot e^4)(e^2 \cdot e^3)u^3,$$

plus a second term multiplying $\Delta(z_1, z_3)\Delta(z_2, z_4)$ which is obtained from (2.19) by switching 2 with 3 and s with u, plus a third term multiplying $\Delta(z_1, z_4)\Delta(z_3, z_2)$ which is obtained from (2.19) by switching 2 with 4 and s with t. Expanding $t_8 F^1 F^2 F^3 F^4$ in terms of polarizations and momenta, one can check that each of these three terms is proportional to $(t_8 F^1 F^2 F^3 F^4)$, and that the sum of the terms is equal to $(t_8 F^1 F^2 F^3 F^4)$ multiplied by

$$c[(t-u)\Delta(z_1, z_2)\Delta(z_3, z_4) + (t-s)\Delta(z_1, z_3)\Delta(z_2, z_4) + (s-u)\Delta(z_1, z_4)\Delta(z_3, z_2)]$$

where c is a constant factor. So it has been proven that the four-point two-loop amplitude computed in [5] coincides with the RNS result of [8][9].

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